

ON THE GENERA OF AN ALMOST SIMPLE GROUP DEFINED OVER AN INTEGRAL DOMAIN OF A GLOBAL FUNCTION FIELD

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ABSTRACT. Let $K = \mathbb{F}_q(C)$ be a global function field arising from a smooth projective curve C defined over a finite field \mathbb{F}_q . The ring of regular functions on $C - S$ where S is any finite set of closed points on C is an integral domain \mathcal{O}_S of K . Given an almost simple \mathcal{O}_S -group \underline{G} with a smooth quasi-split fundamental group \underline{F} , we describe the set of genera of \underline{G} as a product of torsion subgroups of the Brauer groups of finite étale extensions of \mathcal{O}_S . If, furthermore, \underline{G} is not anisotropic of type A_n , its genus bijects to a product of quotients of Picard groups of these extensions. This leads to a necessary and sufficient condition to the Hasse principle to hold for \underline{G} .

1. INTRODUCTION

Let C be a projective algebraic curve defined over a finite field \mathbb{F}_q , assumed to be geometrically connected and smooth. Let $K = \mathbb{F}_q(C)$ be its function field and let Ω denote the set of all closed points of K . For any point $\mathfrak{p} \in \Omega$ let $v_{\mathfrak{p}}$ be the induced discrete valuation on K , $\hat{\mathcal{O}}_{\mathfrak{p}}$ the complete discrete valuation ring w.r.t. $v_{\mathfrak{p}}$ and $\hat{K}_{\mathfrak{p}}$ its fraction field. Any *Hasse set* of K , namely a non-empty finite set $S \subset \Omega$, gives rise to an integral domain of K called an *Hasse domain*:

$$\mathcal{O}_S := \{x \in K : v_{\mathfrak{p}}(x) \geq 0 \ \forall \mathfrak{p} \notin S\}.$$

This is a Dedekind domain, regular and one dimensional. Schemes defined over $\text{Spec } \mathcal{O}_S$ are underlined, being omitted in the notation of their generic fibers.

Let \underline{G} be a semisimple and almost simple \mathcal{O}_S -group with fundamental group \underline{F} , whose cardinality is prime to $\text{char}(K)$. The *genus* $\text{Cl}_S(\underline{G})$ of \underline{G} is the set of (classes of) \underline{G} -torsors that are generically and locally isomorphic to \underline{G} in the flat topology. After relating this term to flat and étale cohomology in Section 2, we shall see in Section 3, that if \underline{F} is quasi-split, namely:

$$\underline{F} \cong \prod_{i=1}^r \text{Res}_{R_i/\mathcal{O}_S}(\underline{\mu}_{m_i}), \quad (1.1)$$

where R_i are finite étale extensions of \mathcal{O}_S , then the *set of genera* of \underline{G} bijects to the abelian group $\prod_{i=1}^r \text{Br}(R_i)[m_i]$. In particular, if \underline{F} is split there are $m^{|S|-1}$ such, where $m := \prod_{i=1}^r m_i$.

Furthermore, if \underline{G} is not anisotropic of type A_n , we show in Section 4, that $\text{Cl}_S(\underline{G})$ bijects as a pointed-set to the abelian group $\prod_{i=1}^r \text{Pic}(R_i)/m_i$. This leads us to assert that the Hasse local-global principle holds for such \underline{G} if and only if any $|\text{Pic}(R_i)|$ is prime to m_i . If \underline{F} is split then this

is just $(|\mathrm{Pic}(\mathcal{O}_S)|, |F|) = 1$. We also refer to the cases of absolutely almost simple groups for which \underline{F} is not quasi-split of types 3,6D_4 or 2E_6 . Finally, an application of the this bijection, towards Tamagawa numbers and local Bruhat-Tits buildings, is exhibited in Section 5.

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2. THE CLASS SET IN ÉTALE COHOMOLOGY

Consider the ring of S -integral adèles $\mathbb{A}_S := \prod_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} \hat{\mathcal{O}}_{\mathfrak{p}}$, being a subring of the adèles \mathbb{A} . The S -class set of \underline{G} is the set of double cosets:

$$\mathrm{Cl}_S(\underline{G}) := \underline{G}(\mathbb{A}_S) \backslash \underline{G}(\mathbb{A}) / G(K)$$

(where for any prime \mathfrak{p} the geometric fiber $\underline{G}_{\mathfrak{p}}$ of \underline{G} is taken). It is finite (cf. [BP, Proposition 3.9]), and its cardinality, called the S -class number of \underline{G} , is denoted by $h_S(\underline{G})$.

By our assumption $(|F|, \mathrm{char}(K)) = 1$, \underline{G} is smooth. Being also affine and finitely generated, it admits according to Nisnevich ([Nis, Theorem I.3.5]) the following exact sequence of pointed sets:

$$1 \rightarrow \mathrm{Cl}_S(\underline{G}) \rightarrow H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}) \rightarrow H^1(K, G) \times \prod_{\mathfrak{p} \notin S} H^1(\hat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}})$$

in which the left exactness reflects the fact that $\mathrm{Cl}_S(\underline{G})$ is the genus of \underline{G} (see its definition in the Introduction). Being also connected, due to Lang's Lemma (recall that all residue fields are finite) this sequence reduces to

$$1 \rightarrow \mathrm{Cl}_S(\underline{G}) \rightarrow H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}) \rightarrow H^1(K, G), \quad (2.1)$$

which indicates that any two \underline{G} -torsors share the same genus if and only if they are K -isomorphic.

Remark 2.1. The classification of \underline{G} -torsors via the étale and flat topologies coincide as \underline{G} is smooth, cf. [SGA4, VIII Cor. 2.3].

Lemma 2.2. *Let \underline{G} be an affine, flat, connected and smooth \mathcal{O}_S -group. Suppose that its generic fiber G is almost simple, simply connected and $\hat{K}_{\mathfrak{p}}$ -isotropic for any $\mathfrak{p} \in S$. Then $H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}) = 1$.*

Proof. The proof, basically relying on the strong approximation property related to G , is the one of Lemma 3.2 in [Bit1], replacing $\{\infty\}$ by S . \square

3. THE SET OF GENERA

The following is the Shapiro's Lemma for the étale cohomology:

Lemma 3.1. *Let $f : R \rightarrow S$ be a finite étale extension of schemes and Γ a smooth R -module. Then $\forall p : H_{\text{ét}}^p(S, \text{Res}_{R/S}(\Gamma)) \cong H_{\text{ét}}^p(R, \Gamma)$.*

Proof. The direct image f_* maps Γ to the induced sheaf, and the spectral sequence $H_{\text{ét}}^p(Y, R^q(f_*\Gamma))$ abuts to $H_{\text{ét}}^{p+q}(X, \Gamma)$ (cf. [Stk, Proposition 54.2]). But as f is finite, $R^q(f_*\Gamma)$ vanishes for any $q \geq 1$ ([Stk, Prop. 55.2(2)]), so this convergence is an equality for $q = 0$: $H_{\text{ét}}^p(S, f_*\Gamma) \cong H_{\text{ét}}^p(R, \Gamma)$. \square

The fundamental group \underline{F} of \underline{G} , being finite, of multiplicative type (cf. [SGA3, XXII, Cor. 4.1.7]), commutative and smooth, is decomposed into finitely many factors of the form $\text{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ or $\text{Res}_{R/\mathcal{O}_S}^{(1)}(\underline{\mu}_m)$ where $\underline{\mu}_m := \text{Spec } \mathcal{O}_S[t]/(t^m - 1)$ and R is some finite étale extension of \mathcal{O}_S .

Proposition 3.2. *Suppose $\underline{F} \cong \prod_{i=1}^r \text{Res}_{R_i/\mathcal{O}_S}(\underline{\mu}_{m_i})$ where R_i are finite étale extensions of \mathcal{O}_S . Then there is an exact sequence of pointed-sets:*

$$1 \rightarrow \text{Cl}_S(\underline{G}) \xrightarrow{h} H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \xrightarrow{w_{\underline{G}}} \prod_{i=1}^r \text{Br}(R_i)[m_i] \rightarrow 1$$

in which h is injective.

Proof. As a pointed set, $\text{Cl}_S(\underline{G})$ is bijective to the first Nisnevich's cohomology set $H_{\text{Nis}}^1(\mathcal{O}_S, \underline{G})$ (cf. [Nis, Theorem I.2.8]), classifying \underline{G} -torsors in the Nisnevich's topology. But Nisnevich's covers are étale, so it is a subset of $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G})$ and h is injective.

Applying étale cohomology to the universal covering of \underline{G} :

$$1 \rightarrow \underline{F} \rightarrow \underline{G}^{\text{sc}} \rightarrow \underline{G} \rightarrow 1. \quad (3.1)$$

gives rise to the coboundary map $\delta_{\underline{G}} : H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \rightarrow H_{\text{ét}}^2(\mathcal{O}_S, \underline{F})$, being surjective as \mathcal{O}_S is of Douai-type, implying that $H_{\text{ét}}^2(\mathcal{O}_S, \underline{G}^{\text{sc}}) = 1$ (see Definition 5.2 and Example 5.4 (iii) in [Gon]).

Assume $\underline{F} = \text{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$. Then Lemma 3.1 with $p = 2$ gives $H_{\text{ét}}^2(\mathcal{O}_S, \underline{F}) \cong H_{\text{ét}}^2(R, \underline{\mu}_m)$. So applying étale cohomology to the related Kummer sequence over R :

$$1 \rightarrow \underline{\mu}_m \rightarrow \underline{\mathbb{G}}_m \xrightarrow{x \mapsto x^m} \underline{\mathbb{G}}_m \rightarrow 1$$

gives rise to the exact sequence of abelian groups:

$$1 \rightarrow \text{Pic}(R)/m \xrightarrow{\partial} H_{\text{ét}}^2(R, \underline{\mu}_m) \xrightarrow{i_*^2} \text{Br}(R)[m] \rightarrow 1, \quad (3.2)$$

in which $\text{Pic}(R)$ is identified with $H_{\text{ét}}^1(R, \underline{\mathbb{G}}_m)$, and the right non-trivial term is the m -torsion part in the Brauer group $\text{Br}(R)$, classifying Azumaya R -algebras and being identified with $H_{\text{ét}}^2(R, \underline{\mathbb{G}}_m)$

(cf. [Mil, §2]). Hence the composition of $\delta_{\underline{G}}$ with i_*^2 is a surjective R -map:

$$w_{\underline{G}} : H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \xrightarrow{\delta_{\underline{G}}} H_{\text{ét}}^2(\mathcal{O}_S, \underline{F}) \cong H_{\text{ét}}^2(R, \underline{\mu}_m) \xrightarrow{(i_*^2)} \text{Br}(R)[m].$$

Since $G^{\text{sc}} := \underline{G}^{\text{sc}} \otimes_{\mathcal{O}_S} K$ is simply connected, $H^1(K, G^{\text{sc}})$ vanishes due to Harder (cf. [Hard, Satz A]), as well as its other K -forms. So Galois cohomology applied to the universal K -covering:

$$1 \rightarrow F \rightarrow G^{\text{sc}} \rightarrow G \rightarrow 1, \quad (3.3)$$

yields an embedding of pointed-sets $\delta_G : H^1(K, G) \hookrightarrow H^2(K, F)$, which is also surjective as K is of Douai-type. Let $K' := K[R]$. Then Galois cohomology applied to the Kummer exact sequence of K' -groups

$$1 \rightarrow \mu_m \rightarrow \mathbb{G}_m \xrightarrow{x \mapsto x^m} \mathbb{G}_m \rightarrow 1$$

yields, together with Hilbert 90 Theorem, the identification $(i_*^2)_{K'} : H^2(K', \mu_m) \cong \text{Br}(K')[m]$, whence the composition of δ_G with $(i_*^2)_{K'}$ is an injective K' -map:

$$w_G : H^1(K, G) \hookrightarrow H^2(K, F) \cong H^2(K', \mu_m) \xrightarrow{(i_*^2)_{K'}} \text{Br}(K')[m].$$

According to Grothendieck, $\text{Br}(R)$ injects into $\text{Br}(K')$ (cf. [Gro, Prop. 2.1] and [Mil, Example 2.22, case (a)]). All together we retrieve the following exact and commutative diagram:

$$\begin{array}{ccc} H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) & \xrightarrow{w_{\underline{G}}} & \text{Br}(R)[m] \\ \downarrow & & \downarrow \\ H^1(K, G) & \xrightarrow{w_G} & \text{Br}(K')[m], \end{array} \quad (3.4)$$

by which we see that $\text{Cl}_S(\underline{G}) \stackrel{(2.1)}{=} \ker[H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \rightarrow H^1(K, G)] = \ker(w_{\underline{G}})$.

More generally, suppose $\underline{F} \cong \prod_{i=1}^r \text{Res}_{R_i/\mathcal{O}_S}(\underline{\mu}_{m_i})$ (see (1.1)). As the cohomology sets commute with direct products, the target groups of $w_{\underline{G}}$ and w_G become $\prod_{i=1}^r \text{Br}(R_i)[m_i]$ and $\prod_{i=1}^r \text{Br}(K'_i)[m_i]$, respectively, so the same argument gives the assertion. \square

Definition 1. The *set of genera* of \underline{G} is the set of isomorphism classes of torsors of \underline{G} modulo K -isomorphisms: $\text{gen}(\underline{G}) := H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) / (H_1 \sim_K H_2)$ (thus being finite).

In the following we adopt the notation of types as appears in [PR, Chap.6].

Corollary 3.3. *If $\underline{F} \cong \prod_{i=1}^r \text{Res}_{R_i/\mathcal{O}_S}(\underline{\mu}_{m_i})$, then there is a bijection of pointed sets:*

$$\text{gen}(\underline{G}) \cong \prod_{i=1}^r \text{Br}(R_i)[m_i].$$

Hence if \underline{F} is split, there are $m^{|S|-1}$ genera, where $m = \prod_{i=1}^r m_i$.

If \underline{G} is absolutely almost simple of types ${}^3\text{D}_4$ or ${}^2\text{E}_6$, then $\text{gen}(\underline{G}) = H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) = 1$.

Proof. The injectivity of h in Proposition 3.2 holds true for the genus of any choice of a base point in $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G})$, thus the latter pointed-set is the disjoint union of all genera of \underline{G} . The obtained short sequence indicates that any two \underline{G} -forms share the same genus if and only if they are mapped by $w_{\underline{G}}$ into the same class in $\prod_{i=1}^r \text{Br}(R_i)[m_i]$.

In particular if \underline{F} is split, then $\text{gen}(\underline{G}) \cong \prod_{i=1}^r \text{Br}(\mathcal{O}_S)[m_i]$. It is shown in the proof of Lemma 2.2 in [Bit1] that $\text{Br}(\mathcal{O}_S) = \ker \left[\mathbb{Q}/\mathbb{Z} \xrightarrow{\sum_{\mathfrak{p} \in S} \text{Cor}_{\mathfrak{p}}} \mathbb{Q}/\mathbb{Z} \right]$ where $\text{Cor}_{\mathfrak{p}}$ is the corestriction map at a prime \mathfrak{p} . Consequently, the cardinality of its m_i -torsion part is $m_i^{|S|-1}$ for any i . The second assertion follows.

If \underline{G} is absolutely almost-simple of types ${}^{3,6}D_4$ or 2E_6 , then the generic fiber F is a subgroup of $\text{Res}_{L/K}^{(1)}(\mu_m)$ where L is finite Galois over K such that $([L : K], m) = 1$ (cf. [PR, p. 332,333]). So we may assume that \underline{F} , being representable and smooth, maintains the same form and property for an étale extension R of \mathcal{O}_S , i.e., fits into an exact sequence of smooth groups:

$$1 \rightarrow \underline{F} \rightarrow \text{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \xrightarrow{N_{R/\mathcal{O}_S}} \underline{\mu}_m \rightarrow 1$$

which yields, again due to Shapiro's Lemma, the short exact sequence:

$$1 \rightarrow \text{coker}(f) \rightarrow H_{\text{ét}}^2(\mathcal{O}_S, \underline{F}) \rightarrow \ker(g) \rightarrow 1 \quad (3.5)$$

where:

$$f : H_{\text{ét}}^1(R, \underline{\mu}_m) \rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mu}_m) \quad \text{and} \quad g : H_{\text{ét}}^2(R, \underline{\mu}_m) \rightarrow H_{\text{ét}}^2(\mathcal{O}_S, \underline{\mu}_m).$$

But as $[R : \mathcal{O}_S]$ is prime to m , R cannot trivialize non-trivial elements of $H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mu}_m)$ and $H_{\text{ét}}^2(\mathcal{O}_S, \underline{\mu}_m)$, which means that f and g are isomorphisms, and so sequence (3.5) implies that $H_{\text{ét}}^2(\mathcal{O}_S, \underline{F}) = 1$. Then étale cohomology applied to the universal cover (3.1) of \underline{G} , yields the exactness of:

$$H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{sc}}) \rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \rightarrow 1.$$

As \underline{G} is not of type A_n , its generic fiber is locally isotropic everywhere (see [BT3, 4.3 and 4.4]), thus according to Lemma 2.2 $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{sc}})$ vanishes, as well as $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G})$. \square

Remark 3.4. As C is smooth, $\text{Spec } \mathcal{O}_S$ is normal, i.e., is integrally closed locally everywhere, thus any finite étale covering of \mathcal{O}_S arises by its normalization in some separable unramified extension of K (see [Len, Theorem 6.13]). Consequently, the condition $\underline{F} \cong \prod_{i=1}^r \text{Res}_{R_i/\mathcal{O}_S}(\underline{\mu}_{m_i})$ appearing in Proposition 3.2 and Corollary 3.3, may be replaced by the one imposed on its generic fiber, namely: $F \cong \prod_{i=1}^r \text{Res}_{L_i/K}(\mu_{m_i})$ where L_i are Galois and unramified over K .

The following table refers to \mathcal{O}_S -groups whose generic fibers are absolutely almost simple groups and adjoint. The missing types from the full list are 2A_n and 2D_n for which the number of genera depends on the particular splitting extensions. The two left columns can be found in page 332 of [PR]. The right one is Corollary 3.3:

Type of G	F	# genera(\underline{G})
${}^1A_{n-1}$	μ_n	$n^{ S -1}$
B_n, C_n, E_7	μ_2	$2^{ S -1}$
${}^{3,6}D_4$	$R_{L/K}^{(1)}(\mu_2)$	1
${}^1D_{n \neq 4}$	$\mu_4, n = 2k + 1$ $\mu_2 \times \mu_2, n = 2k$	$4^{ S -1}$
1E_6	μ_3	$3^{ S -1}$
2E_6	$R_{L/K}^{(1)}(\mu_3)$	1
E_8, F_4, G_2	1	1

Example 3.5. Given a regular quadratic \mathcal{O}_S -space (V, q) of rank $n \geq 3$, its special orthogonal group $\underline{\mathbf{SO}}_q$ is of type A_1 or B_n if $\text{rank}(V)$ is odd, and of type 1D_n otherwise. In both cases the fundamental group of $\underline{\mathbf{SO}}_q$ is μ_2 , thus $\text{Br}(\mathcal{O}_S)[2]$ bijects to the set of $2^{|S|-1}$ proper genera of q (compare with [Bit2, Proposition 4.5]), though for ranks 3, 4 $\underline{\mathbf{SO}}_q$ may be anisotropic.

Corollary 3.6. Any affine smooth \mathbb{F}_q -curve C^{af} is obtained by removing one closed point ∞ from a projective smooth one. Thus any absolutely almost simple group (including of types 2A_n and 2D_n), defined over the ring of regular functions: $\mathcal{O}_S = \mathbb{F}_q[C^{af}]$ ($S = \{\infty\}$), whose fundamental group is split, may posses only one genus.

4. THE GENUS OF LOCALLY ISOTROPIC GROUPS

In this section we assume \underline{G} is not anisotropic of type A_n . As aforementioned, this restriction implies that for any prime \mathfrak{p} , $G_{\mathfrak{p}} := G \otimes_K K_{\mathfrak{p}}$ is $\hat{K}_{\mathfrak{p}}$ -isotropic. Hence the universal covering G^{sc} of G admits the strong approximation property, whence $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{sc}}) = 1$ by Lemma 2.2.

Theorem 4.1. If $\underline{F} \cong \prod_{i=1}^r \text{Res}_{R_i/\mathcal{O}_S}(\underline{\mu}_{m_i})$, then there is a bijection of pointed sets:

$$Cl_S(\underline{G}) \cong \prod_{i=1}^r \text{Pic}(R_i)/m_i.$$

Proof. As aforementioned, $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{sc}})$ vanishes, as well as its generic fiber $H^1(K, G^{\text{sc}})$ due to Harder. Hence applying étale cohomology to sequence (3.1), and Galois cohomology to its generic

fiber (3.3), together with the exactness of sequence 2.1 we get the exact and commutative diagram:

$$\begin{array}{ccccc}
 1 & \longrightarrow & \mathrm{Cl}_S(\underline{G}) & \xrightarrow{g} & \ker(h) \\
 & & \downarrow i & & \downarrow \\
 1 & \longrightarrow & H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}) & \xrightarrow{\delta_{\underline{G}}} & H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, \underline{F}) \\
 & & \downarrow f & & \downarrow h \\
 1 & \longrightarrow & H^1(K, G) & \xrightarrow{(\delta_{\underline{G}})_K} & H^2(K, F)
 \end{array} \tag{4.1}$$

in which the surjectivity of the connecting maps $\delta_{\underline{G}}$ and $(\delta_{\underline{G}})_K$ derives from the fact that \mathcal{O}_S and K are of Douai type, which means that $H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, \underline{G}^{\mathrm{sc}}) = H^2(K, G^{\mathrm{sc}}) = 1$. Given $[\gamma] \in \ker(h)$, the diagram's commutativity shows that $(\delta_{\underline{G}})_K(f(\delta_{\underline{G}}^{-1}([\gamma]))) = [0]$. But $\ker((\delta_{\underline{G}})_K) = 1$, thus $f(\delta_{\underline{G}}^{-1}([\gamma])) = [0]$, i.e., $\delta_{\underline{G}}^{-1}([\gamma]) \in \mathrm{Cl}_S(\underline{G})$ and $g(\delta_{\underline{G}}^{-1}([\gamma])) = [\gamma]$. As a result we get an exact sequence of pointed sets:

$$1 \rightarrow \mathfrak{K}_1 \rightarrow \mathrm{Cl}_S(\underline{G}) \xrightarrow{g} \ker(h) \rightarrow 1.$$

For the injectivity of g , we shall need the assumption that \underline{G} is locally isotropic:

Let \underline{H} be a twisted form of \underline{G} , and let $\underline{H}^{\mathrm{sc}}$ be its universal covering. The lower row in the following exact diagram is the one obtained when replacing the base point \underline{G} by \underline{H} , as described in [Gir, Cha. IV, Proposition 4.3.4]:

$$\begin{array}{ccccccc}
 \mathfrak{K}_1 & \longrightarrow & \mathrm{Cl}_S(\underline{G}) & \xrightarrow{g} & \ker(h) & & \\
 & & \subset & & \downarrow & & \\
 H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}^{\mathrm{sc}}) & \longrightarrow & H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}) & \xrightarrow{\delta_{\underline{G}}} & H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, \underline{\mu}_2) & & \\
 & & \cong \uparrow & & \cong \uparrow & & \\
 H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{H}^{\mathrm{sc}}) & \longrightarrow & H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{H}) & \xrightarrow{\delta_{\underline{G}}} & H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, \underline{\mu}_2) & &
 \end{array} \tag{4.2}$$

As $[\underline{H}] \in \mathrm{Cl}_S(\underline{G})$, it is K -isomorphic to \underline{G} , being thus also K -isotropic, whence $\hat{K}_{\mathfrak{p}}$ -isotropic everywhere, as well as its generic fiber $\underline{H}^{\mathrm{sc}}$. Hence $H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{H}^{\mathrm{sc}})$ is trivial by Lemma 2.2. This is true for any $[\underline{H}] \in \mathrm{Cl}_S(\underline{G})$, which means that $\delta_{\underline{G}}$ restricted to $\mathrm{Cl}_S(\underline{G})$ is an embedding, and so we may conclude that $g : \mathrm{Cl}_S(\underline{G}) \cong \ker(h)$ is a bijection of pointed sets.

Together with sequence (3.2) we get the exact and commutative diagram of abelian groups:

$$\begin{array}{ccccccc}
 & & \mathrm{Cl}_S(\underline{G}) \cong \ker(h) & & & & \\
 & & \downarrow & & & & \\
 1 \longrightarrow & \prod_{i=1}^r \mathrm{Pic}(R_i)/m_i & \longrightarrow & H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, \underline{F}) = H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, \underline{F})^{(i_*^2)^r} & \longrightarrow & \prod_{i=1}^r \mathrm{Br}(R_i)[m_i] & \longrightarrow 1 \\
 & \downarrow & & \downarrow h & & \downarrow & \\
 & 1 & \longrightarrow & H^2(K, F) = H^2(K, F) & \xrightarrow{\cong} & \prod_{i=1}^r \mathrm{Br}(K_i)[m_i] & \longrightarrow 1
 \end{array}$$

from which by the Snake Lemma we may deduce the required bijection. \square

Definition 2. We say that the *local-global Hasse principle* holds for \underline{G} if $h_S(\underline{G}) = 1$.

Corollary 4.2. *The Hasse principle holds for \underline{G} (not anisotropic of type A_n), if and only if $(|Pic(R_i)|, m_i) = 1$ for all i . If \underline{F} splits, then it is just $(|Pic(\mathcal{O}_S)|, |F|) = 1$. This principle holds for any absolutely almost simple group \underline{G} of type ${}^{3,6}D_4$ or 2E_6 .*

Proof. The two first assertion follow from Theorem 4.1 and the third from Corollary 3.3. \square

5. AN APPLICATION TOWARDS THE TAMAGAWA NUMBER AND LOCAL BUILDINGS

Let G be a semi-simple group defined over $K = \mathbb{F}_q(C)$. The Tamagawa number $\tau(G)$ of G is defined as the covolume of the group $G(K)$ in the adelic group $G(\mathbb{A})$ (embedded diagonally as a discrete subgroup), with respect to the Tamagawa measure (see [Weil]). Suppose G is almost simple, not anisotropic of type A_n (still $(|F|, \text{char}(K)) = 1$). Though $\tau(G)$ is defined as a global invariant, in the sequel we shall see that if G and F are quasi-split, $\tau(G)$ can be described as a local invariant, due to the strong approximation property related to the universal covering G^{sc} , and the Weil conjecture, stating that $\tau(G^{\text{sc}}) = 1$, as been recently proved in the function field case by Gaistgory and Lurie (see [Lur, (2.4)]).

We remove one arbitrary closed point ∞ from C and refer as above to the integral domain $\mathcal{O}_{\{\infty\}} = \mathcal{O}_S$. Patching the Bruhat-Tits $\mathcal{O}_{\mathfrak{p}}$ -models of $G_{\mathfrak{p}}$ for all $\mathfrak{p} \neq \infty$ along the generic fiber, results in an affine and smooth $\mathcal{O}_{\{\infty\}}$ -model \underline{G} of G (see [BK, §5]). Let $\mathbb{A}_{\infty} := \hat{K}_{\infty} \times \prod_{\mathfrak{p} \neq \infty} \hat{\mathcal{O}}_{\mathfrak{p}}$ be a subring of \mathbb{A} . Then $\underline{G}(\mathbb{A}_{\infty})G(K)$ is a normal subgroup of $\underline{G}(\mathbb{A})$ (cf. [Tha, Thm. 3.2 3]) and its finite index is $h_{\infty}(G)$. Consider the natural epimorphism:

$$\varphi : \underline{G}(\mathbb{A})/G(K) \rightarrow \underline{G}(\mathbb{A})/\underline{G}(\mathbb{A}_{\infty})G(K) : \forall x \in G(\mathbb{A}) : xG(K) \mapsto x\underline{G}(\mathbb{A}_{\infty})G(K).$$

Since all fibers of φ are isomorphic to $\ker(\varphi)$, we get a bijection of measure spaces

$$\begin{aligned} \underline{G}(\mathbb{A})/G(K) &\cong (\underline{G}(\mathbb{A})/\underline{G}(\mathbb{A}_{\infty})G(K)) \times (\underline{G}(\mathbb{A}_{\infty})/\underline{G}(\mathbb{A}_{\infty}) \cap G(K)) \\ &\cong \text{Cl}_{\infty}(G) \times (\underline{G}(\mathbb{A}_{\infty})/\Gamma) \end{aligned} \tag{5.1}$$

on which the first factor cardinality is $h_{\infty}(G) := h_{\{\infty\}}(\underline{G})$ and $\Gamma := \underline{G}(\mathbb{A}_{\infty}) \cap G(K) = \underline{G}(\mathcal{O}_{\{\infty\}})$.

Applying the Tamagawa measure τ on these spaces we get the Main Theorem in [BK]:

Theorem 5.1. *Let $\mathfrak{g}_\infty = \text{Gal}(\hat{K}_\infty^s/\hat{K}_\infty)$ be the absolute group, $F_\infty := \ker[G_\infty^{sc} \rightarrow G_\infty]$ (defined over \hat{K}_∞), $\underline{F} := \ker[\underline{G}^{sc} \rightarrow \underline{G}]$ and $\widehat{F}_\infty := \text{Hom}(F_\infty \otimes \hat{K}_\infty^s, \mathbb{G}_{m, \hat{K}_\infty^s})$. Then:*

$$\tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)},$$

where: $t_\infty(G) = |\widehat{F}_\infty^{\mathfrak{g}_\infty}|$ is the number of types in one orbit of a special vertex in the Bruhat–Tits building of $G_\infty(\hat{K}_\infty)$ and:

$$j_\infty(G) = \frac{|H_{\text{ét}}^1(\mathcal{O}_{\{\infty\}}, \underline{F})|}{|\underline{F}(\mathcal{O}_{\{\infty\}})|}.$$

Lemma 5.2. *If G and F are quasi-split, then: $h_\infty(G) = j_\infty(G)$.*

Proof. Suppose $\underline{F} = \text{Res}_{R/\mathcal{O}_{\{\infty\}}}(\underline{\mu}_m)$ where R is étale over $\mathcal{O}_{\{\infty\}}$. Let T be a maximal torus in G and let T^{sc} be its preimage in the universal covering G^{sc} . The separable K -isogeny $\pi : T^{\text{sc}} \rightarrow T$ (recall that $(|F|, \text{char}(K)) = 1$) admits an extension over $\text{Spec } \mathcal{O}_{\{\infty\}}$: $\underline{\pi} : \underline{T}^{\text{sc}} \rightarrow \underline{T}^0$ where \underline{T}^0 stands for the connected component of \underline{T} (see p. 11 in [BK]). Since \underline{G} is quasi-split and almost simple, both \underline{T} and $\underline{T}^{\text{sc}}$ are quasi-trivial of the form $\text{Res}_{R/\mathcal{O}_{\{\infty\}}}(\underline{\mathbb{G}}_m^d)$, and so étale cohomology applied to $\underline{\pi}$ together with the Shapiro’s Lemma give rise to the long exact sequence:

$$1 \rightarrow \underline{F}(\mathcal{O}_{\{\infty\}}) \rightarrow \underline{T}^{\text{sc}}(\mathcal{O}_{\{\infty\}}) \xrightarrow{\underline{\pi}} \underline{T}^0(\mathcal{O}_{\{\infty\}}) \rightarrow H_{\text{ét}}^1(R, \underline{\mu}_m) \rightarrow \prod_i \text{Pic}(R) \xrightarrow{[\mathcal{L}] \rightarrow [\mathcal{L}^{\otimes m}]} \prod_i \text{Pic}(R). \quad (5.2)$$

Since C^{af} lacks only one point from a projective curve, the invertible elements in $\mathcal{O}_{\{\infty\}}$ are units, which means that $\underline{T}^{\text{sc}}(\mathcal{O}_{\{\infty\}}) = \underline{T}^{\text{sc}}(\mathbb{F}_q)$ and $\underline{T}^0(\mathcal{O}_{\{\infty\}}) = \underline{T}^0(\mathbb{F}_q)$. As $\underline{T}^{\text{sc}}$ and \underline{T}^0 are smooth, these rational \mathbb{F}_q -points are captured by their specializations $\overline{T}^{\text{sc}} := \underline{T}^{\text{sc}} \otimes_{\text{Spec } \mathcal{O}_{\{\infty\}}} \text{Spec } \mathbb{F}_q$ and $\overline{T}^0 := \underline{T}^0 \otimes_{\text{Spec } \mathcal{O}_{\{\infty\}}} \text{Spec } \mathbb{F}_q$ where $\text{Spec } \mathbb{F}_q \hookrightarrow \text{Spec } \mathcal{O}_{\{\infty\}}$ is the closed immersion of the special point, and so the exact sequence (5.2) can be rewritten as:

$$1 \rightarrow \underline{F}(\mathcal{O}_{\{\infty\}}) \rightarrow \overline{T}^{\text{sc}}(\mathbb{F}_q) \xrightarrow{\bar{\pi}} \overline{T}^0(\mathbb{F}_q) \rightarrow H_{\text{ét}}^1(R, \underline{\mu}_m) \rightarrow \text{Pic}(R)[m] \rightarrow 1. \quad (5.3)$$

The isogenous schemes \overline{T}^{sc} and \overline{T}^0 are connected and defined over a finite field, thus sharing the same number of \mathbb{F}_q -points (see [Bor, §16.8]), so the exactness of (5.3) implies:

$$|\underline{F}(\mathcal{O}_{\{\infty\}})| = \frac{|H_{\text{ét}}^1(\mathcal{O}_{\{\infty\}}, \underline{F})|}{|\text{Pic}(R)[m]|}.$$

As $\text{Pic}(R)$ is a finite abelian group, the right hand denominator is equal to $|\text{Pic}(R)/m|$, being equal according to Theorem 4.1 to $h_{\{\infty\}}(\underline{G})$. More generally, if $F \cong \prod \text{Res}_{R_i/\mathcal{O}_{\{\infty\}}}(\underline{\mu}_{m_i})$, this direct product commutes with the cohomology sets, and the assertion follows. \square

Theorem 5.1 and Lemma 5.2 imply:

Corollary 5.3. *Let G be an almost simple semisimple group, not anisotropic of type A_n defined over a global function field K , being quasi-split as well as its fundamental group. Its Tamagawa number $\tau(G)$ is a local invariant: $|\widehat{F_\infty}^{\mathfrak{g}_\infty}|$, being equal to the number of types in one orbit of a special vertex in the Bruhat–Tits building of $G_\infty(\hat{K}_\infty)$, for any choice of a closed point ∞ on C .*

Remark 5.4. By the geometric version of Čebotarev’s density theorem (see in [Jar]), one may choose the point ∞ such that G_∞ is split, whence $\tau(G) = |F_\infty|$.

REFERENCES

- [SGA4] M. Artin, A. Grothendieck, J.-L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas* (SGA 4) LNM, Springer, 1972/1973.
- [Bor] A. Borel, *Linear Algebraic Groups*, Benjamin, New York (1969).
- [BK] R. A. Bitan, R. Kohl *A building-theoretic approach to relative Tamagawa numbers of semisimple groups over global function fields*, *Funct. Approx. Comment. Math.* **53**, Number 2, 2015, 215–247.
- [Bit1] R. A. Bitan, *The Hasse principle for bilinear symmetric forms over a ring of integers of a global function field*, *J. Number Theory*, 2016, 346–359.
- [Bit2] R. A. Bitan, *On the classification of quadratic forms over an integral domain of a global function field*, preprint.
- [BT3] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local. III: Compléments et applications à la cohomologie galoisienne*, *J. Fac. Sci. Univ. Tokyo* **34** (1987), 671–688.
- [BP] A. Borel, G. Prasad, *Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups*, *Publ. Math. IHES* **69** (1989), 119–171.
- [Gir] J. Giraud, *Cohomologie non abélienne*, Grundlehren math. Wiss., Springer-Verlag Berlin Heidelberg New York, 1971.
- [SGA3] M. Demazure, A. Grothendieck, *Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes*, Tome II, Réédition de SGA3, P. Gille, P. Polo, 2011.
- [Gon] C. D. González-Avilés, *Quasi-abelian crossed modules and nonabelian cohomology*, *J. of Algebra* **369**, 2012, 235–255.
- [Gro] A. Grothendieck, *Le groupe de Brauer III: Exemples et compléments*, *Dix Exposés sur la Cohomologie des Schémas*, North-Holland, Amsterdam, 1968, 88–188.
- [Hard] G. Harder, *Über die Galoiscohomologie halbeinfacher algebraischer Gruppen, III*, *J. Reine Angew. Math.* **274/275**, 1975, 125–138.
- [Jar] M. Jarden, *The Čebotarev density theorem for function fields: An elementary approach*, *Math. Ann.*, 261 **4** (1982), 467–475.
- [KP] J. Kluners, S. Pauli, *Computing residue class rings and Picard groups of orders*, *J. Algebra* **292** (2005), 47–64.
- [Len] H. W. Lenstra, *Galois theory for schemes*, <http://websites.math.leidenuniv.nl/algebra/GSchemes.pdf>
- [Lur] J. Lurie *Tamagawa Numbers of Algebraic Groups Over Function Fields*.
- [Mil] J. S. Milne, *Étale Cohomology*, Princeton University Press, Princeton, 1980.
- [Nis] Y. Nisnevich, *Étale Cohomology and Arithmetic of Semisimple Groups*, PhD thesis, Harvard University, 1982.
- [PR] V. Platonov, A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, San Diego 1994.
- [Ser] J.-P. Serre, *Cohomologie Galoisienne*, *Lecture Notes in Mathematics*, Vol. 5, Springer-Verlag, New York, 1965.
- [Stk] <http://stacks.math.columbia.edu/download/etale-cohomology.pdf>
- [Tha] N. Q. Thang, *A Norm Principle for class groups of reductive group schemes over Dedekind rings*, *Vietnam J. Math.*, June 2015, **43**, Issue 2, 257–281.
- [Weil] A. Weil, *Adèles and Algebraic Groups*, *Progress in Mathematics*, Birkhauser, Basel, 1982.